Topologia
Lista 5

Zad 1. Prove that a topological space (X, τ) is compact if and only if it satisfies *Riesz condition*, that is when: for every family $\{F_t\}_{t \in T}$ of closed sets

$$\bigcap_{t \in T} F_t = \emptyset \implies \exists_{\{t_1, \dots, t_n\} \subset T} \; F_{t_1} \cap \dots \cap F_{t_n} = \emptyset.$$

Zad 2. Which of the sets X (and A_i in item a)) from Exercise 11 on List 3 are compact.

Zad 3. Show that an image of a compact space under continuous mapping is a compact space.

Zad 4. Show that every closed subset of a compact space is compact.

Zad 5. Prove that if X a Hausdorff space, then a compact subset of X is closed. Is this statement true for spaces not satisying the T_2-axiom?

Zad 6. Show that any (infinite) sequence of points in compact Hausdorff space has an accumulation point.

Zad 7. Show that a compact metric space is complete.

Zad 8. Prove that in a compact metric space (X, ρ) for every $\varepsilon > 0$ there exists *ε-net*, that is a finite set of points $A_\varepsilon = \{q_1, \dots, q_k\}$ such that

$$\rho(x, A_\varepsilon) < \varepsilon, \qquad \text{for all } x \in X.$$

Zad 9. Show that a compact metric space is bounded.

Zad 10. Prove that a compact metric space is separable.

Zad 11. Show that a topological space with countable basis is separable.

Zad 12. Prove that every metric separable space possess a countable basis

Zad 13. Show that if X is compact, Y is Hausdorff space and $f : X \to Y$ is a continuous surjection, then

i) f is a closed mapping, that is an image of a closed set under f is again closed,

ii) if f is injective then it is a homeomorphism.

Zad 14. *Cantor set* $\mathcal{C}$ is a subset of the unit interval consiting of the numbers of the following form

$$t = \frac{t_1}{3} + \frac{t_2}{3^2} + \frac{t_3}{3^3} + \dots, \qquad t_n \in \{0, 2\}.$$

Show that Cantot set is closed, has no isolated points and its boundary is empty.

Zad 15. Show that a Cantor set is homeomorphic to infinite cartesian product of the discrete space of power two:

$$\mathcal{C} \cong \{0, 2\} \times \{0, 2\} \times \{0, 2\} \times \dots .$$